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NONLINEAR FREE VIBRATIONS OF ELASTIC STRUCTURES

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Abstract—An approach to nonlinear free vibrations of elastic structures is developed with the aid of Hamilton's principle and a perturbation procedure. The theory is analogous to the theory of initial postbuckling behavior due to Koiter. It provides information regarding the first order effects of finite displacements upon the frequency, period and dynamic stresses arising in the free, undamped vibration of structures. Attention is restricted to structures which are linearly elastic. The theory is illustrated by application to the free vibration of beams and rectangular plates.

INTRODUCTION

A THEORY of initial postbuckling behavior has been developed by Koiter [1, 2] which permits the first order effects of finite displacements and initial imperfections on the buckling process to be assessed. Although the original work was based upon potential energy considerations, Budiansky and Hutchinson [3] and Budiansky [4] rederived the essentials of the theory by another method which is based upon virtual work. In the present paper an approach is developed for the analysis of nonlinear free vibrations that is in much the same spirit. The approach is quite analogous to the treatment of Koiter's theory in Ref. [4].

A perturbation approach that is applicable to nonlinear partial differential equations which possess periodic solutions has been outlined by Keller [5] and applied by Keller and Ting [6]. The essential features of the perturbation approach are used in the present development. Solutions to the governing equations are sought as a power series in the amplitude of the linear vibration mode and higher order effects are systematically generated by successive perturbation equations.

OUTLINE OF THE THEORY

To facilitate a concise presentation of the theory, the functional notation used by Budiansky [4] will be employed. The motion of the structure produces generalized displacement $\mathbf{u}$, strain $\boldsymbol{\gamma}$ and stress $\boldsymbol{\sigma}$. The dynamics of the system is established by Hamilton's principle, which is symbolically written

$$\int_0^{2\pi/\omega} \left[\delta\left(\frac{1}{2}M\left(\frac{\partial \mathbf{u}}{\partial t}\right) \cdot \frac{\partial \mathbf{u}}{\partial t}\right) - \boldsymbol{\sigma} \cdot \delta\boldsymbol{\gamma} \right] \mathrm{d}t = 0. \tag{1}$$

The "dot" operation signifies the appropriate inner multiplication of variables and integration of the result over the entire structure. The generalized mass operator M is assumed to be homogeneous and linear with the property that

$$M(\mathbf{u}) \, . \, \mathbf{v} = M(\mathbf{v}) \, . \, \mathbf{u} \tag{2}$$

for all $\mathbf{u}$ and $\mathbf{v}$. Since only periodic motion is to be considered, the limits on the integral over time correspond to a single period of the motion. ω is the circular frequency of the vibration such that

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{u}\left(\mathbf{r}, t+\frac{2\pi}{\omega}\right) \tag{3}$$

$\mathbf{r}$ is the position vector to an arbitrary point in the structure.

If a new time variable $\tau = \omega t$ is introduced in equation (1), this equation may be replaced by

$$\int_0^{2\pi} [\omega^2 \delta(\tfrac{1}{2}M(\dot{\mathbf{u}}) \, . \, \dot{\mathbf{u}}) - \boldsymbol{\sigma} \, . \, \boldsymbol{\delta\gamma}] \, \mathrm{d}\tau = 0. \tag{4}$$

In the above, the notation $(\dot{\ })= \partial(\ )/\partial\tau$ has been used. An integration by parts results in

$$\omega^2 M(\dot{\mathbf{u}}) \, . \, \boldsymbol{\delta}\mathbf{u}|_0^{2\pi} - \int_0^{2\pi} [\omega^2 M(\ddot{\mathbf{u}}) \, . \, \boldsymbol{\delta}\mathbf{u} + \boldsymbol{\sigma} \, . \, \boldsymbol{\delta\gamma}] \, \mathrm{d}\tau = 0. \tag{5}$$

The boundary terms vanish by reason of periodicity and we are left with

$$\int_0^{2\pi} [\omega^2 M(\ddot{\mathbf{u}}) \, . \, \boldsymbol{\delta}\mathbf{u} + \boldsymbol{\sigma} \, . \, \boldsymbol{\delta\gamma}] \, \mathrm{d}\tau = 0 \tag{6}$$

$\boldsymbol{\delta}\mathbf{u}$ is any virtual displacement that is consistent with all the kinematic boundary conditions imposed on the structure.

Equation (6) is supplemented by the strain–displacement relation

$$\boldsymbol{\gamma} = L_1(\mathbf{u}) + \tfrac{1}{2}L_2(\mathbf{u}) \tag{7}$$

where L_1 and L_2 are homogeneous linear and quadratic functionals, respectively. In addition, the homogeneous bilinear functional L_{11} is defined by the following equation:

$$L_2(\mathbf{u}+\mathbf{v}) = L_2(\mathbf{u}) + 2L_{11}(\mathbf{u}, \mathbf{v}) + L_2(\mathbf{v}). \tag{8}$$

It follows that $L_{11}(\mathbf{u}, \mathbf{v}) = L_{11}(\mathbf{v}, \mathbf{u})$ and $L_{11}(\mathbf{u}, \mathbf{u}) = L_2(\mathbf{u})$. If use is made of the above definitions, the variation of the generalized strain can be written as

$$\boldsymbol{\delta\gamma} = \boldsymbol{\delta}\mathbf{e} + L_{11}(\mathbf{u}, \boldsymbol{\delta}\mathbf{u}) \tag{9}$$

where

$$\mathbf{e} = L_1(\mathbf{u}) \tag{10}$$

is the linearized strain measure.

For Hookean (linear) elastic structures, the stress–strain relation can be written in the form

$$\boldsymbol{\sigma} = H(\boldsymbol{\gamma}) \tag{11}$$

where H is a homogeneous linear function. The following reciprocity relation

$$\boldsymbol{\sigma}^{(1)} \, . \, \boldsymbol{\gamma}^{(2)} = \boldsymbol{\sigma}^{(2)} \, . \, \boldsymbol{\gamma}^{(1)} \tag{12}$$

will be assumed also; "1" and "2" are any arbitrary states of stress and strain.

The vibration modes and frequencies of the linearized theory can be found by setting

$$\mathbf{u} = \xi \mathbf{u}_1 \qquad \boldsymbol{\gamma} = \xi \mathbf{e}_1 \qquad \boldsymbol{\sigma} = \xi \boldsymbol{\sigma}_1 \tag{13}$$

ξ is an amplitude parameter associated with the mode $\mathbf{u}_1$ which has natural frequency ω_0. If equation (13) is substituted into equation (6) and only linear terms are retained, we obtain

$$\int_0^{2\pi} [\omega_0^2 M(\ddot{\mathbf{u}}_1) \,.\, \boldsymbol{\delta}\mathbf{u} + \boldsymbol{\sigma}_1 \,.\, \boldsymbol{\delta}\mathbf{e}] \, d\tau = 0. \tag{14}$$

Equating the integrand to zero yields the linearized equation of motion.

If we now set $\boldsymbol{\delta}\mathbf{u} = \mathbf{u}_1$ in the above equation, we obtain an expression for ω_0^2.

$$\omega_0^2 = \frac{-\int_0^{2\pi} \boldsymbol{\sigma}_1 \,.\, \mathbf{e}_1 \, d\tau}{\int_0^{2\pi} M(\ddot{\mathbf{u}}_1) \,.\, \mathbf{u}_1 \, d\tau} = \frac{\int_0^{2\pi} \boldsymbol{\sigma}_1 \,.\, \mathbf{e}_1 \, d\tau}{\int_0^{2\pi} M(\dot{\mathbf{u}}_1) \,.\, \dot{\mathbf{u}}_1 \, d\tau}. \tag{15}$$

This is analogous to a Rayleigh quotient for the natural frequency ω_0.

We assume at this point that a single mode $\mathbf{u}_1$ is associated with the natural frequency ω_0. The case of multiple modes corresponding to the same natural frequency will be discussed later.

To discover how the structure behaves for finite amplitudes, we assume

$$\begin{aligned} \mathbf{u} &= \xi \mathbf{u}_1 + \xi^2 \mathbf{u}_2 + \dots \\ \gamma &= \xi \mathbf{e}_1 + \xi^2 (\mathbf{e}_2 + \tfrac{1}{2} L_2(\mathbf{u}_1)) + \dots \\ \boldsymbol{\sigma} &= \xi \boldsymbol{\sigma}_1 + \xi^2 \boldsymbol{\sigma}_2 + \dots \end{aligned} \tag{16}$$

where, in order to make the expansions unique, the displacement increments $\mathbf{u}_2$, $\mathbf{u}_3$, ... are orthogonalized with respect to $\mathbf{u}_1$ in the sense that

$$M(\dot{\mathbf{u}}_1) \,.\, \dot{\mathbf{u}}_k = M(\ddot{\mathbf{u}}_1) \,.\, \mathbf{u}_k = 0 \qquad (k \neq 1). \tag{17}$$

This relation, together with equation (12), also implies that

$$\boldsymbol{\sigma}_1 \,.\, \mathbf{e}_k = 0 \qquad (k \neq 1) \tag{18}$$

which by virtue of the reciprocity relation (12) implies further that

$$H(\mathbf{e}_k) \,.\, \mathbf{e}_1 = 0 \qquad (k \neq 1). \tag{19}$$

The substitution of (16) into (6) yields

$$\begin{aligned} \int_0^{2\pi} \{ & \xi(\omega^2 M(\ddot{\mathbf{u}}_1) \,.\, \boldsymbol{\delta}\mathbf{u} + \boldsymbol{\sigma}_1 \,.\, \boldsymbol{\delta}\mathbf{e}) + \xi^2(\omega^2 M(\ddot{\mathbf{u}}_2) \,.\, \boldsymbol{\delta}\mathbf{u} + \boldsymbol{\sigma}_2 \,.\, \boldsymbol{\delta}\mathbf{e} + \boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \boldsymbol{\delta}\mathbf{u})) \\ & + \xi^3(\omega^2 M(\ddot{\mathbf{u}}_3) \,.\, \boldsymbol{\delta}\mathbf{u} + \boldsymbol{\sigma}_3 \,.\, \boldsymbol{\delta}\mathbf{e} + \boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_2, \boldsymbol{\delta}\mathbf{u}) + \boldsymbol{\sigma}_2 \,.\, L_{11}(\mathbf{u}_1, \boldsymbol{\delta}\mathbf{u})) + \dots \} \, d\tau = 0. \end{aligned} \tag{20}$$

If we now set $\boldsymbol{\delta}\mathbf{u} = \mathbf{u}_1$, $\boldsymbol{\delta}\mathbf{e} = \mathbf{e}_1$ in this expression and introduce (15), the following result is obtained:

$$\begin{aligned} \int_0^{2\pi} \Bigg\{ & \xi\left(1 - \frac{\omega^2}{\omega_0^2}\right) \boldsymbol{\sigma}_1 \,.\, \mathbf{e}_1 + \xi^2(\boldsymbol{\sigma}_2 \,.\, \mathbf{e}_1 + \boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_1)) \\ & + \xi^3(\boldsymbol{\sigma}_3 \,.\, \mathbf{e}_1 + \boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_2) + \boldsymbol{\sigma}_2 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_1)) + \dots \Bigg\} \, d\tau = 0. \end{aligned}$$

However, the reciprocity relation (12) permits further simplification as

$$\boldsymbol{\sigma}_2 \,.\, \mathbf{e}_1 = \boldsymbol{\sigma}_1 \,.\, \boldsymbol{\gamma}_2 = \boldsymbol{\sigma}_1 \,.\, (\mathbf{e}_2 + \tfrac{1}{2}L_{11}(\mathbf{u}_1, \mathbf{u}_1)) \tag{21}$$

$$= \tfrac{1}{2}\boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_1)$$

and

$$\boldsymbol{\sigma}_3 \,.\, \mathbf{e}_1 = \boldsymbol{\sigma}_1 \,.\, \boldsymbol{\gamma}_3 = \boldsymbol{\sigma}_1 \,.\, (\mathbf{e}_3 + L_{11}(\mathbf{u}_1, \mathbf{u}_2)) \tag{22}$$

$$= \boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_2).$$

Consequently,

$$\int_0^{2\pi} \left\{ \xi\left(1 - \frac{\omega^2}{\omega_0^2}\right)\boldsymbol{\sigma}_1 \,.\, \mathbf{e}_1 + \xi^2(\tfrac{3}{2}\boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_1)) + \xi^3(2\boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_2) + \boldsymbol{\sigma}_2 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_1)) + \dots \right\} \mathrm{d}\tau = 0 \tag{23}$$

and we have the asymptotic relation

$$\frac{\omega^2}{\omega_0^2} = 1 + A\xi + B\xi^2 + \dots \tag{24}$$

where

$$A = \frac{\int_0^{2\pi} \frac{3}{2}\boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_1)\mathrm{d}\tau}{\int_0^{2\pi} \boldsymbol{\sigma}_1 \,.\, \mathbf{e}_1 \,\mathrm{d}\tau} \tag{25}$$

$$= \frac{\int_0^{2\pi} \frac{3}{2}\boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_1)\,\mathrm{d}\tau}{\omega_0^2 \int_0^{2\pi} M(\dot{\mathbf{u}}_1) \,.\, \dot{\mathbf{u}}_1 \,\mathrm{d}\tau}$$

and

$$B = \frac{\int_0^{2\pi} (2\boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \mathbf{u}_2) + \boldsymbol{\sigma}_2 \,.\, L(\mathbf{u}_1, \mathbf{u}_1))\,\mathrm{d}\tau}{\omega_0^2 \int_0^{2\pi} M(\dot{\mathbf{u}}_1) \,.\, \dot{\mathbf{u}}_1 \,\mathrm{d}\tau}. \tag{26}$$

If A is nonzero, the structure can exhibit a softening characteristic with the frequency decreasing for finite amplitudes for $A\xi$ negative. If A is zero, a negative value of B corresponds to softening (decreasing frequency) and a positive, nonzero value corresponds to hardening (increasing frequency). If both A and B are zero, higher order terms must be investigated to discover the nature of finite amplitude effects.

In the evaluation of B a solution for $\mathbf{u}_2$ and $\boldsymbol{\sigma}_2$ is required. It must be found from the second order term in equation (20). The variational equation of motion is, therefore,

$$\omega^2 M(\ddot{\mathbf{u}}_2) \,.\, \boldsymbol{\delta}\mathbf{u} + \boldsymbol{\sigma}_2 \,.\, \boldsymbol{\delta}\mathbf{e} + \boldsymbol{\sigma}_1 \,.\, L_{11}(\mathbf{u}_1, \boldsymbol{\delta}\mathbf{u}) = 0 \tag{27}$$

and

$$\boldsymbol{\gamma}_2 = \mathbf{e}_2 + \tfrac{1}{2}L(\mathbf{u}_1, \mathbf{u}_1) \qquad \boldsymbol{\sigma}_2 = H(\boldsymbol{\gamma}_2)$$

$\boldsymbol{\delta}\mathbf{u}$ is orthogonal to $\mathbf{u}_1$ in the sense of equation (17).

MULTIPLE MODE SITUATIONS

If more than one mode corresponds to the same natural frequency found from the linearized theory, the above described solution process requires modification. If we assume that k modes correspond to the same frequency, with the linearly independent modes being identified as $\mathbf{u}_{11}, \mathbf{u}_{12}, \ldots, \mathbf{u}_{1k}$, then the expansion for displacement in equation (16) must be replaced by

$$\mathbf{u} = \sum_{i=1}^{k} \xi_i \mathbf{u}_{1i} + \mathbf{w} \tag{28}$$

where the modes are orthogonalized with respect to each other and to the additional displacement $\mathbf{w}$. Also we write

$$\boldsymbol{\sigma} = \sum_{i=1}^{k} \xi_i \boldsymbol{\sigma}_{1i} + \mathbf{s}. \tag{29}$$

It is possible to develop k equations of the same type as (23) which are obtained by setting $\delta\mathbf{u}$ equal to $\mathbf{u}_{11}, \mathbf{u}_{12}, \ldots, \mathbf{u}_{1k}$. These equations, retaining only quadratic terms in the ξ_i's as was done by Budiansky and Hutchinson [3] and thus neglecting the effects of $\mathbf{w}$, must be solved simultaneously. Since the equations are homogeneous, only amplitude ratios can be found. A higher order analysis can be made, but the solution process is far more difficult than the case of a single vibration mode.

APPLICATION TO FLEXURAL VIBRATIONS OF BEAMS

Consider the uniform, simply supported beam shown in Fig. 1. An axial force is induced in the beam due to finite amplitude vibrations because the supports are assumed to be immovable. Let U and W be the longitudinal and transverse components of displacement, N be the axial force and X and Z be the coordinates shown in the figure. The average axial strain ε is given by

$$\varepsilon = U_{,X} + \tfrac{1}{2}(W_{,X})^2 = \frac{N}{EA}, \tag{30}$$

where E is Young's modulus and A is the cross-sectional area. Since the supports cannot move apart

$$U(L) - U(0) = \int_0^L U_{,X}\, \mathrm{d}X = 0 \tag{31}$$

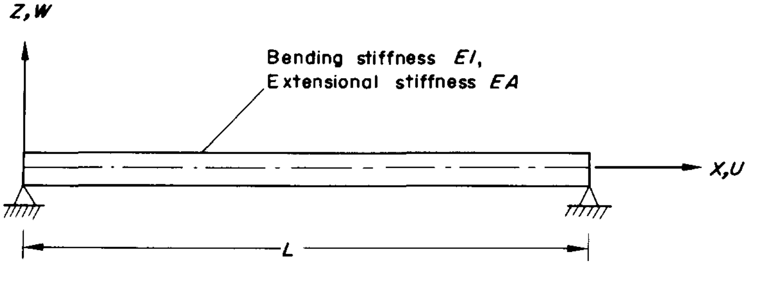

FIG. 1.

which implies

$$N = \frac{EA}{2L}\int_0^L (W_{,X})^2\,dX. \tag{32}$$

The equation governing transverse vibrations of the beam is

$$mW_{,tt} + EIW_{,XXXX} - NW_{,XX} = 0, \tag{33}$$

where m is the mass per unit length of the beam and I is the second moment of area. If we introduce the following variables and parameters,

$$n = \frac{NL^2}{\pi^2 EI} \qquad x = \frac{\pi X}{L} \qquad w = \frac{W}{\sqrt{(\pi\rho)}} \qquad \rho = \sqrt{\left(\frac{I}{A}\right)} \qquad \omega_1^2 = \frac{\pi^4 EI}{mL^4} \tag{34}$$

then equations (32) and (33) can be written in a convenient dimensionless form. They become

$$n = \frac{1}{2}\int_0^\pi (w_{,x})^2\,dx \tag{35}$$

$$\frac{1}{(\omega_1)^2}w_{,tt} + w_{,xxxx} - nw_{,xx} = 0. \tag{36}$$

Equation (36) is subject to the boundary conditions

$$\begin{aligned} w(0,t) &= w_{,xx}(0,t) = 0 \\ w(\pi,t) &= w_{,xx}(\pi,t) = 0. \end{aligned} \tag{37}$$

We now set $\tau = \omega t$, $\Omega = \omega^2/\omega_1^2$, and, in view of the structure of the equations,

$$\begin{aligned} w &= \xi w_1 + \xi^3 w_3 + \dots \\ n &= \xi^2 n_2 + \xi^4 n_4 + \dots. \end{aligned} \tag{38}$$

The first approximation involves only w_1. The linearized equation is

$$\Omega \ddot{w}_1 + w_{1,xxxx} = 0. \tag{39}$$

The solution is of the form

$$w_1 = \cos\tau \sin kx, \tag{40}$$

where k is an integer. This solution yields

$$\Omega_0 = k^4 \tag{41}$$

for the dimensionless frequency parameter.

The second approximation is simply

$$n_2 = \frac{1}{2}\int_0^\pi (w_{1,x})^2\,dx = \frac{\pi k^2}{8}(1+\cos 2\tau). \tag{42}$$

The coefficient A in the expansion (24) is zero. B is determined to be

$$B = \frac{\int_0^{2\pi}\int_0^\pi n_2(w_{1,x})^2\,dx\,d\tau}{\Omega_0\int_0^{2\pi}\int_0^\pi (\dot{w}_1)^2\,dx\,d\tau} = \frac{3\pi}{16}. \tag{43}$$

Consequently, we have the asymptotic relation

$$\frac{\Omega}{\Omega_0} = 1+\frac{3\pi}{16}\xi^2+\dots \tag{44}$$

This asymptotic approximation can be shown to be in complete agreement with an asymptotic representation of the solution obtained by Woinowsky–Krieger [7] in terms of elliptic functions and with the solutions obtained by Chu and Herrmann [8] using two different methods.

APPLICATION TO FLEXURAL VIBRATIONS OF RECTANGULAR PLATES

A rectangular plate of length a, width b and thickness h is shown in Fig. 2, along with notation and a sign convention. The origin of surface coordinates (X, Y) is taken to be the corner of the plate, and U, V and W are midsurface displacement components. The plate is assumed to be simply supported on all four edges and relative motions of the edges are assumed to be prevented. Under these conditions, membrane stresses will be induced in the plate due to transverse flexural vibrations of finite amplitude.

The analysis is based upon the following equations, which are equivalent to the dynamic version of von Karman's equations used by Chu and Herrmann [8]:

$$\nabla^4 f = (w_{,xy})^2 - w_{,xx}w_{,yy} \tag{45}$$

$$\frac{4}{(\omega_{sp})^2}w_{,tt}+\nabla^4 w - f_{,yy}w_{,xx}+2f_{,xy}w_{,xy}-f_{,xx}w_{,yy} = 0 \tag{46}$$

$$u_{,x} = f_{,yy}-\nu f_{,xx}-\tfrac{1}{2}(w_{,x})^2 \tag{47}$$

$$v_{,y} = f_{,xx}-\nu f_{,yy}-\tfrac{1}{2}(w_{,y})^2. \tag{48}$$

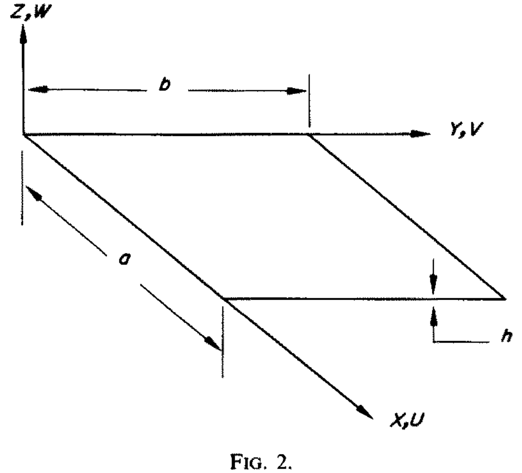

FIG. 2.

The dimensionless variables and parameters are listed below, along with any other important quantities.

$$x = \frac{\pi X}{b} \qquad y = \frac{\pi Y}{b} \qquad w = \frac{[12(1-\nu^2)]^{\frac{1}{2}} W}{h} \qquad u = \frac{4\pi E}{\sigma_p b} U \qquad v = \frac{4\pi E}{\sigma_p b} V$$

$$(\omega_{sp})^2 = \frac{Eh^3}{3(1-\nu^2)m}\left(\frac{\pi}{b}\right)^4 \qquad \mu = \frac{b}{a} \qquad \sigma_p = \frac{E}{3(1-\nu^2)}\left(\frac{\pi h}{b}\right)^2$$

$$\nabla^2(\ ) = \frac{\partial^2(\ )}{\partial x^2} + \frac{\partial^2(\ )}{\partial y^2} \qquad \nabla^4(\ ) = \nabla^2[\nabla^2(\ )]. \tag{49}$$

The stress parameter σ_p is the compressive buckling stress of long or square simply supported plates; ω_{sp} is the fundamental circular frequency of a square plate. ν is Poisson's ratio and f is a dimensionless Airy stress function defined such that the membrane stresses, σ_x^0, σ_y^0, and τ_{xy}^0, are given by the relation

$$\{\sigma_x^0, \sigma_y^0, \tau_{xy}^0\} = \frac{\sigma_p}{4}\{f_{,yy}, f_{,xx}, -f_{,xy}\}. \tag{50}$$

The above partial differential equations are subject to the following boundary conditions:

$$w(0, y) = w\left(\frac{\pi}{\mu}, y\right) = w(x, 0) = w(x, \pi) = 0 \tag{51}$$

$$w_{,xx}(0, y) = w_{,xx}\left(\frac{\pi}{\mu}, y\right) = w_{,yy}(x, 0) = w_{,yy}(x, \pi) = 0 \tag{52}$$

$$v(x, 0) = v(x, \pi) = 0 \tag{53}$$

$$u(0, y) = u\left(\frac{\pi}{\mu}, y\right) = 0. \tag{54}$$

$\mu = b/a$ is the aspect ratio of the plate.

We set $\tau = \omega t$, $\Omega = \dfrac{\omega^2}{(\omega_{sp})^2}$ and

$$\begin{aligned} w &= \xi w_1 + \xi^3 w_3 + \ldots \\ f &= \xi^2 f_2 + \xi^4 f_4 + \ldots. \end{aligned} \tag{55}$$

The first order equation involves only w_1 and is

$$4\Omega \ddot{w}_1 + \nabla^4 w_1 = 0. \tag{56}$$

We take the solution corresponding to the fundamental mode of the plate in the form

$$w_1 = \sin \mu x \sin y \sin \tau, \tag{57}$$

which leads to the linearized frequency ratio

$$\Omega_0 = \frac{(\mu^2+1)^2}{4}. \tag{58}$$

The second order equation is

$$\nabla^4 f_2 = (w_{1,xy})^2 - w_{1,xx}w_{1,yy} = \frac{\mu^2(1-\cos 2\tau)}{4}(\cos 2\mu x + \cos 2y). \tag{59}$$

We take the following solution for f_2:

$$f_2 = \tfrac{1}{64}(1-\cos 2\tau)\left[\frac{1}{\mu^2}\cos 2\mu x + \mu^2 \cos 2y + 2\alpha x^2 + 2\beta y^2\right] \tag{60}$$

α and β are constants which must be evaluated so as to satisfy the boundary conditions (53) and (54). The tangential displacement parameters can be determined with f_2 and w_1 known; they are

$$u_2 = \frac{1}{32\mu}(1-\cos 2\tau)\sin 2\mu x(\nu - \mu^2 + \mu^2 \cos 2y) \tag{61}$$

$$v_2 = \tfrac{1}{32}(1-\cos 2\tau)\sin 2y(\nu\mu^2 - 1 + \cos 2\mu x). \tag{62}$$

It can be verified that these expressions satisfy (53) and (54) and that we must require that

$$\alpha = \frac{1+\nu\mu^2}{(1-\nu^2)} \tag{63}$$

and

$$\beta = \frac{(\mu^2+\nu)}{(1-\nu^2)}. \tag{64}$$

Again, as for the beam, the coefficient A in the expansion (24) is zero. For the plate, B is found to be

$$B = \frac{\int_0^{2\pi}\int_0^{\pi/\mu}\int_0^{\pi}[f_{2,yy}(w_{1,x})^2 + f_{2,xx}(w_{1,y})^2]\,dx\,dy\,d\tau}{4\Omega_0\int_0^{2\pi}\int_0^{\pi/\mu}\int_0^{\pi}(\dot{w})^2\,dx\,dy\,d\tau} \tag{65}$$

$$= \frac{3}{32(\mu^2+1)}\left[\frac{\mu^2+2\nu\mu^2+1}{(1-\nu^2)} + \frac{1}{2}(\mu^4+1)\right].$$

Consequently, we have the asymptotic relation

$$\frac{\Omega}{\Omega_0} \cong 1 + \frac{3}{32(\mu^2+1)}\left[\frac{\mu^2+2\nu\mu^2+1}{(1-\nu^2)} + \frac{1}{2}(\mu^4+1)\right]\xi^2. \tag{66}$$

A solution to this problem has been obtained by Chu and Herrmann [8] in terms of elliptic functions. A careful study of their solution and an asymptotic representation of it for small amplitudes shows that there is complete agreement between it and the present solution, albeit a lengthy process.

CONCLUDING REMARKS

A general approach to nonlinear vibrations of elastic structures has been developed which provides information regarding the first order effects of finite displacements. It relies upon the use of Hamilton's principle and a perturbation procedure to obtain analytical results. The theory effectively reduces a nonlinear free vibration problem to a sequence of linear problems, only the first two of which usually need be solved to obtain an initial estimate of finite amplitude effects. The theory has been applied to beams and rectangular plates, structures for which solutions already exist, in order to illustrate the theory and demonstrate its usefulness.

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Абстракт—С помощью принципа Гамильтона и метода возмущений, выводится подход к решению нелинейных, свободных колебаний упругих конструкций. Теория аналогична к теории начального закритического поведения в смысле Койтера. Дает она информацию, касающуюся эффектов первого рода для конечных перемещений на частоту, период и динамические напряжения, возникающие во время свободного, незатухающего колебания конструкций. Ограничивается внимание к линейно-упругим конструкциям. Для иллюстрации, теория применяется к задаче свободного колебания балок и прямоугольных пластинок.